PAPER

The O-Sequence: Representation of 3D-Dissection

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SUMMARY A 3D-dissection (A rectangular solid dissection) is a dissection of a rectangular solid into smaller rectangular solids by planes. In this paper, we propose an O-sequence, a string of representing any 3D-dissection which is dissected by only non-crossing rectangular planes. We also present a necessary and sufficient condition for a given string to be an O-sequence.

key words: *rectangular solid dissection, rectangular dissection, 3D-VLSI, 2D-dissection, 3D-dissection, Q-sequence, O-sequence*

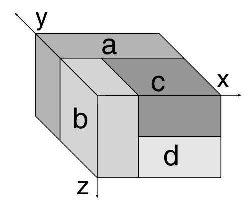

Fig. 1 3D-dissection restricted to a slicing structure.

1. Introduction

Recently, a 3D-packing and a rectangular solid dissection (3D-dissection) have attracted attention, and have been studied much. They will be usable in various fields, e.g. layout design of 3D-VLSI, scheduling of dynamically reconfigurable processor and so on.

A 3D-packing is an arrangement of rectangular solids (blocks) into a rectangular solid box without overlapping each other. Several representations of a 3D-packing have been proposed [1]–[4], and it is proved that some of them can represent any 3D-packing.

On the other hand, a 3D-dissection is a dissection of a rectangular solid (entire rectangular solid) into smaller rectangular solids (rooms) by planes (dissection walls). Every dissection wall is parallel to one of the faces of the entire rectangular solid. The information about positions of walls will be usable for some applications. For example, in 3D-VLSI layout designs, a design method of wiring modules (blocks in rooms) along walls will be effective, which is similar to a method of wiring modules in channels in 2D-VLSI layout designs.

Lei et al. represent 3D-dissections restricted to a slicing structure by a slicing tree [5]. The slicing structure is obtained by slicing one region into two regions recursively. 3D-dissections restricted to a slicing structure are dissected by rectangular dissection walls. Figure 1(a) shows a 3D-dissection restricted to a slicing structure.

Ma et al. proposed 3D-CBL [6] as a representation of a 3D-dissection which is dissected by only rectangular and non-crossing walls*. Since any 3D-dissection restricted to a slicing structures is dissected by only rectangular and non-crossing walls, a set of 3D-dissection which is dissected by only rectangular and non-crossing walls contains any 3D-dissection restricted to a slicing structure.

3D-CBL can represent any 3D-dissection which is dissected by only rectangular and non-crossing walls. However, there is not an one-to-one corresponding relation between a 3D-CBL and a 3D-dissection. So, several codes of 3D-CBL are decoded into the same 3D-dissection. In addition, several other codes are not decoded into a 3D-dissection.

In this paper, we propose a new method representing a 3D-dissection by a string called O-sequence. O-sequence represents any 3D-dissection which is dissected by only rectangular and non-crossing walls. There is an one-to-one corresponding relation between an O-sequence and a 3D-dissection which is dissected by only rectangular and non-crossing walls. We also present a necessary and sufficient condition for a given string to be an O-sequence. In addition, we present an algorithm for decoding an O-sequence to a 3D-dissection in $O(n)$ time.

The rest of the paper is organized as follows: In Sect. 2, we explain 2D-dissection and Q-Sequence, and present some theorems. In Sect. 3, we explain 3D-dissection which is dissected by only rectangular and non-crossing walls, and propose O-Sequence, and present a necessary and sufficient condition for a given string to be an O-sequence. In Sect. 4, a size of the solution space of O-Sequence will be shown. In Sect. 5, experimental results will be shown. Finally, Sect. 6

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*Ma et al. insist that 3D-CBL can represent any 3D-dissection if it has no empty rooms and no crossing walls. However, there are 3D-dissections which are dissected by non-rectangular walls, and can't be represented by 3D-CBL in fact. For example, a rectangular solid dissection shown in Fig. 2(a) can't be represented by 3D-CBL.

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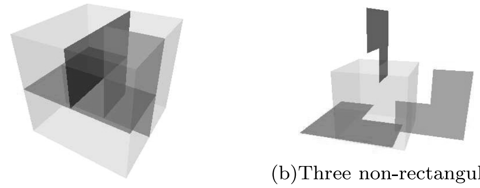

(a)A 3D-dissection into five rooms (b)Three non-rectangular dissection walls of (a)

Fig. 2 A 3D-dissection of general structure which is dissected by three non-rectangular dissection walls into five rooms.

gives our conclusion.

2. Rectangular Dissection and Q-Sequence

In this section, we explain the Q-sequence, a method of representing a 2D-dissection (a rectangular dissection), proposed by Sakanushi et al. [7], and show an important property of Q-sequences.

Consider a rectangle, called the *entire rectangle*, which consists of the left, right, top, and bottom segments (segs). A 2D-dissection with n rooms is dissected by $n-1$ internal segs. A vertical seg and a horizontal seg are not permitted to cross each other, but can touch to form a T-shaped junction (T-junction), if at least one of two segs is internal seg. In the following, we focus on the right-bottom and left-top corners of the entire rectangle. If both a right seg and a bottom seg of a room are sides of the entire rectangle, the room is a right-bottom room and is denoted by RB. If both a left seg and a top seg of a room are sides of the entire rectangle, the room is the left-top room and is denoted by LT. For any room r except RB, two segs meet at the right-bottom corner of r and one ends there forming a T-junction. The seg that ends at the T-junction is called the *prime seg* of r. Note that RB does not have a prime seg.

Rooms that are adjacent to prime seg of r on the opposite side of r are called the *associated rooms* of r. If prime seg of r is a vertical (horizontal) seg, the topmost (leftmost) of the associated rooms is called the *next room* of r. It is known that we can order the rooms as follows: (i) Label LT with 1; (ii) If room r is labeled with i then label the next room of r with $i+1$ (See Fig. 3). It is also known that this ordering is equivalent to the following one: Let F_1 denote the original 2D-dissection, and for any positive integer $i \leq n-1$, let F_{i+1} be the 2D-dissection obtained from F_i by sliding the prime seg of LT to the left or top until LT is deleted. Then, label the room corresponding to LT in F_i with i. This ordering is known as *Abe ordering* [8].

For any positive integer $i \leq n$, let $r(i)$ denote the i-th room in Abe ordering, $Q'(i)$ be the sequence of symbols R (if the prime seg of $r(i)$ is vertical) or B (if the prime seg is horizontal) with subscripts of the associated rooms of $r(i)$ in the decreasing order of Abe ordering, and let $Q(i) = r(i)Q'(i)$ (Define that $Q'(n)$ is the empty sequence). $Q(W_R)[Q(W_B)]$ is the sequence of symbols $R[B]$ with subscripts of the rooms on the left[top] side of the entire rectangle in the decreasing order of Abe ordering. Then, we define the Q-sequence Q of a 2D-dissection as

$$Q = Q(W_R)Q(W_B)Q(1)Q(2)\cdots Q(n).$$

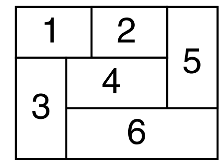

Fig. 3 Abe ordering.

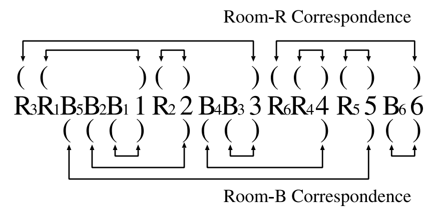

Fig. 4 Parenthesis systems.

The Q-sequence of the rectangular dissection shown in Fig. 3 is $R_3R_1B_5B_2B_11R_22B_4B_33R_6R_44R_55B_66$, where $r(i) = i$. There is one-to-one corresponding relation between a Q-sequence and a rectangular dissection.

We introduce a U-sequence to describe a necessary and sufficient condition for a sequence to be a Q-sequence. the U-sequence is a sequence

$$\mathcal{U} = \mathcal{U}(W_R)\mathcal{U}(W_B)1\mathcal{U}(1)2\mathcal{U}(2)\cdots\mathcal{U}(n-1)n$$

consisting of $1, 2, \ldots, n$, Rs, and Bs such that $\mathcal{U}(W_R)$ $[\mathcal{U}(W_B)]$ is a sequence of Rs$[B$s$]$.

The following theorem is proved in [7]

Theorem 1: [7] A U-sequence $\mathcal{U}$ is a Q-sequence if and only if $\mathcal{U}$ satisfies the following two conditions:

1. $\mathcal{U}(i)$ is a sequence of length at least one, and the sequence of positional symbols consisting exclusively of R or B;
2. Subscripts are assigned to the Rs and Bs so that the sequence forms parenthesis systems under the ordered pairing of (R_k, k), and also under the ordered pairing of (B_k, k) for $k = 1, 2, \ldots, n$ (Fig. 4);

□

By condition 2 in Theorem 1, we can represent a 2D-dissection with

$$Q_{\mathrm{simp}} = Q(1)Q(2)\cdots Q(n)$$

instead of Q-sequence. We call Q_{simp} a simplified Q-sequence.

An algorithm for decoding a given Q-sequence into a corresponding 2D-dissection is shown in Fig. 5. This algorithm performs in $O(n)$ time.

Then, we show an important property of Q-sequences for proving Theorem 4 in Sect. 3.

Theorem 2: The region consisting of rooms $r(1), r(2), \ldots, r(i)$ is a rectangle if and only if for every positive integer $j < i$, $Q'(j)$ contains neither $R_{r(k)}$ nor $B_{r(k)}$ with $k > i$.

Proof : Assume that the region consisting of rooms $r(1), r(2), \ldots, r(i)$ is a rectangle, and assume for contradiction that $Q'(j)$ contains $R_{r(k)}$ for some positive integers j and k with $j < i < k$. Then, by condition 2 in Theorem 1,

$$Q'(j) = \cdots R_{r(k)} \cdots R_{r(j+1)},$$

which means that rooms $r(k)$ and $r(j+1)$ are right of the prime seg of $r(j)$ and on the prime seg of $r(j)$ as shown in Fig. 6. However, the region containing rooms $r(j)$ and $r(j+1)$, but not $r(k)$, can not be a rectangle, which is a contradiction. Hence, $Q'(j)$ does not contain $R_{r(k)}$. Similarly, we can prove that $Q'(j)$ does not contain $B_{r(k)}$.

If $Q'(j)$ contains neither $R_{r(k)}$ nor $B_{r(k)}$ with $k > i$ for every positive integer $j < i$ then all of $r(1), r(2), \ldots, r(i-1)$ are inserted into the region in which $r(i)$ is inserted by decoding algorithm for a Q-sequence. Hence, the region consisting of rooms $r(1), r(2), \ldots, r(i)$ is a rectangle. □

It is easy to obtain the following theorem from the decoding algorithm in Fig. 5.

Theorem 3: Let

$$Q = Q(r(1))Q(r(2)) \cdots Q(r(n)),$$

be the simplified Q-sequence of a 2D-dissection F, and assume that the region consisting of rooms $r(1), r(2), \ldots, r(i)$ is a rectangle in the 2D-dissection. Then, the simplified Q-sequence of the 2D-dissection obtained from F by replacing rooms $r(1), r(2), \ldots, r(i)$ with a single room $r(0)$ is

```
Decoding Algorithm
    Input: Q-sequence Q;
    Output: 2D-dissection;
{
  Prepare an entire rectangle dissected
            into only one room, labeled with n;
  for (k = n − 1, n − 2, ..., 1) {
    if (Q(k) = R_{r_m} R_{r_{m−1}} ··· R_{r_1})
      Insert room k from the left of
                    rooms r_m, r_{m−1}, ..., r_1;
    if (Q(k) = B_{r_m} B_{r_{m−1}} ··· B_{r_1})
      Insert room k from the top of
                    rooms r_m, r_{m−1}, ..., r_1;
  }
}
```

Fig. 5 Algorithm for decoding Q-sequence.

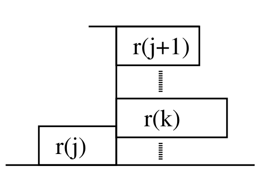

Fig. 6 Region containing rooms $r(j)$ and $r(j+1)$, but not $r(k)$ is not rectangular.

$$Q' = Q(r(0))Q(r(i+1)) \cdots Q(r(n)),$$

where $Q(r(0)) = r(0)Q'(r(i))$. □

3. 3D-Dissection which is Dissected by Rectangular Walls and O-Sequence

Consider a rectangular solid, called the *entire rectangular solid*, which consists of the left, right, front, back, top, and bottom faces. A dissection of the entire rectangular solid, called 3D-dissection for short, is a dissection of the rectangular solid into smaller rectangular solids (called *rooms*) by planes (called *dissection walls*).

Throughout the paper, we assume that a 3D-dissection satisfies the following three conditions:

(i) Every dissection wall is a rectangle;

(ii) Any two dissection walls do not cross each other;

(iii) For any dissection wall w and any line segment l comprising the boundary of w, there exists exactly one wall perpendicular to w and containing l, where a "wall" means a dissection wall or a face of the entire rectangular solid.

Figure 7 shows examples of 3D-dissections satisfying conditions (i), (ii), and (iii). The 3D-dissection in Fig. 7(a) is not a slicing-structure, while that in Fig. 7(b) is a slicing-structure. Notice that any 3D-dissection of a slicing-structure satisfies (i), (ii), and (iii).

In this paper, two 3D-dissections are regarded to be equivalent if one can be obtained from the other by sliding dissection walls without overlapping dissection walls. By this equivalence relation, the 3D-dissections can be classified into a finite number of classes. We will propose a string to each equivalence class.

Dissection wall w is called a *prime wall* of room r if r is on w and if the RBB (right-back-bottom) vertex of r is a vertex of w. By the definition of prime walls, the following Lemma is trivial:

Lemma 1: Every room except the RBB room has exactly one prime wall. □

The rooms that are adjacent to r's prime wall on the opposite side of r are called the *associated rooms* of r.

Let D be any 3D-dissection which is dissected into n rooms. If $n \geq 2$, we can erase the LFT(left-front-top) room in D by sliding the prime wall of the LFT room to the left, front, or top, which results in a new 3D-dissection which

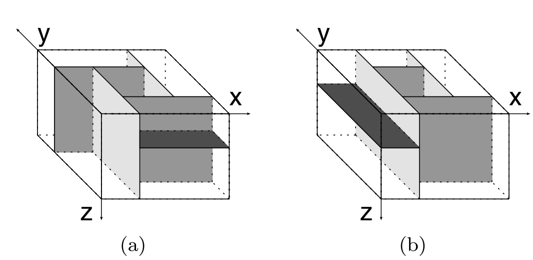

Fig. 7 3D-dissections which is dissected by only non-crossing rectangular walls.

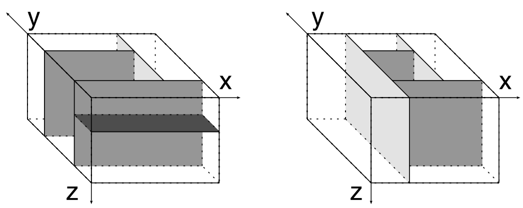

Fig. 8 3D-dissections which are obtained by erasing one LFT room from original 3D-dissections in Fig. 7.

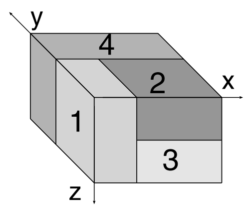

Fig. 9 3D Abe ordering.

is dissected into $n-1$ rooms (See Fig. 8). We can repeat this operation until the number of rooms is one. Then, each room in D can be numbered uniquely in the erased order (The only non-erased room is labeled with n). We call this ordering *3D Abe ordering*.

In this paper, the direction from left[resp. front, top] to right[back, bottom] is viewed as the x[y, z]-direction.

Consider any 3D-dissection into n rooms. For any positive integer $i \le n$, let $r(i)$ denote the i-th room in 3D Abe ordering, $O'(i)$ be the sequence of symbols X (if the prime wall of $r(i)$ is perpendicular to the x-axis), Y (if the prime wall is perpendicular to the y-axis), or Z (if the prime wall is perpendicular to the z-axis) with subscripts of the associated rooms of $r(i)$ in the decreasing order of 3D Abe ordering, and let $O(i) = r(i)O'(i)$ (Define that $O'(n)$ is the empty sequence). $O(W_X)$[resp. $O(W_Y)$ and $O(W_Z)$] is the sequence of symbols X[Y and Z] with subscripts of the rooms on the left[front and top] face in the decreasing order of 3D Abe ordering.

The O-sequence O of the 3D-dissection is defined as the concatenation of $O(W_X), O(W_Y), O(W_Z), O(1), O(2), \ldots$, and $O(n)$, that is

$$O = O(W_X)O(W_Y)O(W_Z)O(1)O(2)\cdots O(n).$$

There is an one-to-one corresponding relation between an O-sequence and a 3D-dissection which is dissected by only rectangular and non-crossing walls. The O-sequence of the 3D-dissection shown in Fig. 9 is $X_4X_1Y_3Y_2Y_1 Z_4Z_2Z_1 1 X_3X_2 2 Z_3 3 Y_4 4$, where $r(i) = i$.

Next, we introduce the P-sequence to describe a necessary and sufficient condition for a sequence to be an O-sequence. A P-sequence is a sequence

$$\mathcal{P} = \mathcal{P}(W_X)\mathcal{P}(W_Y)\mathcal{P}(W_Z)1\mathcal{P}(1)2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n$$

consisting of $1, 2, \ldots, n$, Xs, Ys, and Zs such that $\mathcal{P}(W_X)$ [$\mathcal{P}(W_Y)$, $\mathcal{P}(W_Z)$] is a sequence of Xs[Ys, Zs].

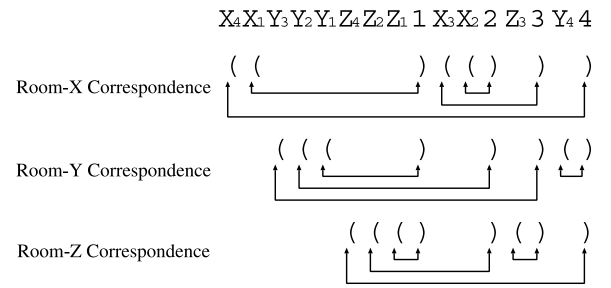

Fig. 10 Parenthesis systems.

Theorem 4: A P-sequence $\mathcal{P}$ is an O-sequence if and only if $\mathcal{P}$ satisfies the following three conditions:

1. $\mathcal{P}(i)$ is a sequence of length at least one of positional symbols consisting exclusively of X, Y, or Z;
2. Subscripts are assigned to the Xs, Ys and Zs so that the sequence forms parenthesis systems under the ordered pairing of (X_k, k), under the ordered pairing of (Y_k, k), and also under the ordered pairing of (Z_k, k) for $k = 1, 2, \ldots, n$ (Fig. 10);
3. Let $\mathcal{P}(i) = X_{r_m}\cdots X_{r_2}X_{r_1}$ [resp. $\mathcal{P}(i) = Y_{r_m}\cdots Y_{r_1}$, $\mathcal{P}(i) = Z_{r_m}\cdots Z_{r_1}$]. For any integer l with $r_m < l \le n$, at least one of the following two conditions is satisfied:
   a. $\mathcal{P}(r_1), \ldots, \mathcal{P}(r_{m-1})$ contain neither Y_l nor Z_l [Z_l nor X_l, X_l nor Y_l];
   b. Either of $\mathcal{P}(r_m), \ldots, \mathcal{P}(l-1)$ contains X_l[Y_l, Z_l].

□

3.1 Proof of Theorem 4

In 3.1.1, the necessity of conditions 1, 2, and 3 in Theorem 4 will be given in Theorem 5, 6, and 8, respectively. In 3.1.2, the sufficiency of Theorem 4 will be proved constructively.

3.1.1 Proof of Necessity of Theorem 4

Assume that P-sequence $\mathcal{P}$ is an O-sequence of a 3D-dissection. By the definition of O-sequence, the following theorem is trivial:

Theorem 5: Condition 1 in Theorem 4 is satisfied. □

It is easy to see that the last characters of $\mathcal{P}(W_X)$, $\mathcal{P}(W_Y)$, and $\mathcal{P}(W_Z)$ are X_1, Y_1, and Z_1, respectively. $\mathcal{P}(W_X)$, $\mathcal{P}(W_Y)$, and $\mathcal{P}(W_Z)$ can be represented as $\mathcal{P}'(W_X)X_1$, $\mathcal{P}'(W_X)Y_1$ and $\mathcal{P}'(W_Z)Z_1$, respectively.

Lemma 2: Let $n \ge 2$. If $\mathcal{P}(1)$ consists of Xs[resp. Ys and Zs] then the sequence

$$\mathcal{P}' = (\mathcal{P}'(W_X)\mathcal{P}(1))\mathcal{P}'(W_Y)\mathcal{P}'(W_Z)2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n$$
$$[\mathcal{P}' = \mathcal{P}'(W_X)(\mathcal{P}'(W_Y)\mathcal{P}(1))\mathcal{P}'(W_Z)2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n$$
and
$$\mathcal{P}' = \mathcal{P}'(W_X)\mathcal{P}'(W_Y)(\mathcal{P}'(W_Z)\mathcal{P}(1))2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n]$$

is the O-sequence of the 3D-dissection obtained from the original 3D-dissection by sliding the prime wall of room 1 until room 1 is deleted.

Proof : Let F be the 3D-dissection represented by $\mathcal{P}$. Let F' be a 3D-dissection obtained from F by sliding the prime wall of room 1 until room 1 is deleted, and let

$$\mathcal{P}'' = \mathcal{P}''(W_X)\mathcal{P}''(W_Y)\mathcal{P}''(W_Z)2\mathcal{P}''(2)\cdots\mathcal{P}''(n-1)n$$

be the O-sequence of F'. For any integer i with $2 \le i \le n-1$, the associated rooms of room i in F is also that in F', and so $\mathcal{P}''(i) = \mathcal{P}(i)$. Assume that $\mathcal{P}(1)$ consists of Xs, that is

$$\mathcal{P}(1) = X_{i_k}X_{i_{k-1}}\ldots X_{i_1}.$$

Then, it is easy to see that $\mathcal{P}''(W_Y) = \mathcal{P}'(W_Y)$ and $\mathcal{P}''(W_Z) = \mathcal{P}'(W_Z)$. Moreover, the rooms $i_1, \ldots, i_k$ are on the left face in F' because these rooms are associated rooms of room 1 in F, that is on the prime wall of room 1 in F. Let j_Y and j_Z be rooms such that the prime walls of j_Y and j_Z are back and bottom on room 1, respectively. Assume without loss of generality that $j_Y < j_Z$. Then, $i_1, \ldots, i_k \le j_Y$ since rooms $i_1, \ldots, i_k$ are left to the prime wall of j_k, and must be deleted before room j_Y is deleted. On the other hand, if X_l is in $\mathcal{P}'(W_X)$ then room l is not deleted before room j_Y because l is back of the prime wall of room j_Y or bottom of that of room j_Z. That is, $l \ge j_Y$. Thus, we conclude that $\mathcal{P}''(W_X) = \mathcal{P}'(W_X)\mathcal{P}(1)$, and hence $\mathcal{P}'' = \mathcal{P}'$. □

Theorem 6: Condition 2 in Theorem 4 is satisfied.

Proof : The theorem is proved by induction on n. The theorem is true for $n = 1$ trivially.

Assume that the theorem holds for any $n \le k$, and consider the case when $n = k + 1$. Assume without loss of generality that $\mathcal{P}(1)$ consists of Xs. By Lemma 2,

$$\mathcal{P}' = (\mathcal{P}'(W_X)\mathcal{P}(1))\mathcal{P}'(W_Y)\mathcal{P}'(W_Z)2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n$$

is the O-sequence of the 3D-dissection obtained from the original 3D-dissection by sliding the prime wall of room 1 until room 1 is deleted. Therefore, $\mathcal{P}'$ satisfies condition 2 by the inductive hypothesis, and hence, $\mathcal{P}$ also satisfies condition 2. □

We can use as an O-sequence

$$\mathcal{P}_{\text{simp}} = 1\mathcal{P}(1)2\mathcal{P}(2)\cdots(n-1)\mathcal{P}(n-1)n$$

instead of $\mathcal{P}$ because $\mathcal{P}$ can be obtained from $\mathcal{P}_{\text{simp}}$ by using Theorem 6. We call $\mathcal{P}_{\text{simp}}$ a simplified O-sequence. Using a simplified O-sequence, Lemma 2 can be described as follows:

Lemma 3: If $\mathcal{P}_{\text{simp}} = 1\mathcal{P}(1)2\mathcal{P}(2)\cdots(n-1)\mathcal{P}(n-1)n$ is a simplified O-sequence then

$$2\mathcal{P}(2)\cdots(n-1)\mathcal{P}(n-1)n$$

is the simplified O-sequence of the 3D-dissection obtained from the original 3D-dissection by sliding the prime wall of room 1 until room 1 is deleted. □

Define that

$$X(\mathcal{P}) = \{i : X_i \in \mathcal{P}(j) \text{ for some } j \in [1, n-1]\},$$
$$Y(\mathcal{P}) = \{i : Y_i \in \mathcal{P}(j) \text{ for some } j \in [1, n-1]\},$$
$$Z(\mathcal{P}) = \{i : Z_i \in \mathcal{P}(j) \text{ for some } j \in [1, n-1]\}.$$

Let $\mathcal{P}_X$[resp. $\mathcal{P}_Y$ and $\mathcal{P}_Z$] denote the sequence obtained from $\mathcal{P}_{\text{simp}}$ by deleting i, X_i, Y_i, and Z_i for every $i \in X(\mathcal{P})$ [$i \in Y(\mathcal{P})$ and $i \in Z(\mathcal{P})$]. For example, $\mathcal{P}_{\text{simp}}$ of the 3D-dissection in Fig. 9 is $1X_3X_22Z_33Y_44$, then $X(\mathcal{P}) = \{2, 3\}$. By deleting 2, 3, X_2, X_3, Y_2, Y_3, Z_2 and Z_3 from $\mathcal{P}_{\text{simp}}$, $\mathcal{P}_X$=$1Y_44$ is obtained. Note that this is the simplified Q-sequence of the 2D-dissection on the left face of the entire rectangular solid in Fig. 9.

Lemma 4: $\mathcal{P}_X$[resp. $\mathcal{P}_Y$ and $\mathcal{P}_Z$] is the simplified Q-sequence of the 2D-dissection on the left[front and top] face of the entire rectangular solid, if we use symbols R and B instead of Y and Z[instead of Z and X, and instead of X and Y].

Proof : The proof is by the induction on n.

The lemma holds for $n = 1$ trivially.

Assume that the lemma holds for any $n \le k$ ($k \ge 1$), and consider the case when $n = k+1$. Let F be the 3D-dissection represented by $\mathcal{P}_{\text{simp}}$, and let F' be a 3D-dissection obtained from F by sliding the prime wall of room 1 until room 1 is deleted. By Lemma 3, the O-sequence of F' is

$$\mathcal{P}'_{\text{simp}} = 2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n.$$

Since F' has $n - 1 = k$ rooms, we obtain, by the inductive hypothesis, the simplified Q-sequence of the 2D-dissection on the left face of the entire rectangular solid in F',

$$\mathcal{P}'_X = j_1\mathcal{P}'(j_1)j_2\mathcal{P}'(j_2)\cdots\mathcal{P}'(j_{m-1})j_m.$$

Assume that $\mathcal{P}(1)$ consists of Xs. Then, the 2D-dissection on the left face of the entire rectangular solid in F is obtained from the 2D-dissection represented by $\mathcal{P}'_X$ by replacing rooms $j_1, j_2, \ldots, j_l$ with a single room 1 for some l, and so is denoted by a Q-sequence

$$Q = 1\mathcal{P}'(j_l)j_{l+1}\mathcal{P}'(j_{l+1})\cdots\mathcal{P}'(j_{m-1})j_m$$

by Theorem 4. On the other hand, we have

$$\mathcal{P}(1) = X_{j_l}X_{j_{l-1}}\cdots X_{j_1}.$$

since the associated rooms of 1 in F are $j_1, j_2, \ldots, j_l$. Hence, we obtain that $\mathcal{P}_X$ is obtained from $\mathcal{P}'_X$ by deleting j, Y_j, and Z_j for $j = j_1, j_2, \ldots, j_l$ and by adding 1 from the left. By Theorem 1 and (Parenthesis systems), we conclude that $Q = \mathcal{P}_X$, and hence $\mathcal{P}_X$ is the 2D-dissection on the left face of the entire rectangular solid in F.

Assume that $\mathcal{P}(1)$ consists of Ys. Then, the 2D-dissection on the left face of the entire rectangular solid in F is obtained from the 2D-dissection represented by $\mathcal{P}'_X$ by inserting a room 1 from the front of rooms $j'_1, j'_2, \ldots, j'_l$, and so is denoted by a Q-sequence

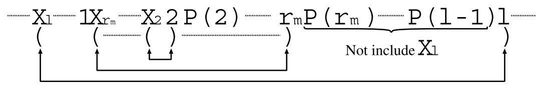

Fig. 11 Position of X_l in $\mathcal{P}$.

$$Q = 1Q(1)j_1\mathcal{P}'(j_1)j_2\mathcal{P}'(j_2)\cdots\mathcal{P}'(j_{m-1})j_m,$$

where $Q(1) = Y_{j'_1}Y_{j'_2}\cdots Y_{j'_{l'}}$. Notice that $j'_1, j'_2, \ldots, j'_{l'}$ are the rooms such that each of the rooms is the associated rooms of room 1 in F and on the front face of the entire rectangular solid. On the other hand, we obtain

$$\mathcal{P}_X = 1\mathcal{P}'(1)j_1\mathcal{P}'(j_1)j_2\mathcal{P}'(j_2)\cdots\mathcal{P}'(j_{m-1})j_m$$

by the definitions of $\mathcal{P}_X$ and $\mathcal{P}'_X$, where $\mathcal{P}'(1)$ is the sequence obtained from $\mathcal{P}(1)$ by deleting i, X_i, Y_i, and Z_i for every $i \in \mathcal{P}_X$. Notice that Y_j is contained in $\mathcal{P}'(1)$ if and only if room i is an associated room of room 1 and is on the left face of the entire rectangular solid. Therefore, we conclude that $Q = \mathcal{P}_X$, and hence $\mathcal{P}_X$ is the 2D-dissection on the left face of the entire rectangular solid in F.

By a similar argument, we can prove that $\mathcal{P}_X$ is the 2D-dissection on the left face of the entire rectangular solid in F in the case when $\mathcal{P}(1)$ consists of Zs. □

Theorem 7: Let i be any integer with $1 \leq i \leq n-1$, and assume that $\mathcal{P}(i) = X_{r_m}\cdots X_{r_1}$[resp. $\mathcal{P}(i) = Y_{r_m}\cdots Y_{r_1}$, and $\mathcal{P}(i) = Z_{r_m}\cdots Z_{r_1}$]. Then, for every integer l, $r_m < l \leq n$, at least one of the following two conditions is satisfied:

1. $\mathcal{P}(i+1), \ldots, \mathcal{P}(r_m - 1)$ contain neither Y_l nor Z_l[Z_l nor X_l, and X_l nor Y_l];
2. $\mathcal{P}(r_m), \ldots, \mathcal{P}(l-1)$ contain X_l[Y_l, and Z_l].

Proof : The theorem is proved by induction on n. The theorem is trivial for $n = 1$.

Assume that the theorem holds for any $n \leq k$, and consider the case when $n = k+1$. By Lemma 2 and the inductive hypothesis, at least one of two conditions is satisfied for every $i \geq 2$. Assume without loss of generality that $\mathcal{P}(1)$ consists of Xs. By Theorem 6, we have $r_1 = 2$. Assume that for some l with $r_m < l \leq n$

- $\mathcal{P}(2), \ldots, \mathcal{P}(r_m - 1)$ contain Y_l or Z_l, and
- $\mathcal{P}(r_m), \ldots, \mathcal{P}(l-1)$ do not contain X_l.

Since $\mathcal{P}$ is a parenthesis system under the ordered pairing of (X_k, k) by Theorem 6, $\mathcal{P}(1), \ldots, \mathcal{P}(r_m - 1)$ do not contain X_l, and hence, we conclude that $\mathcal{P}(1), \ldots, \mathcal{P}(l-1)$ do not contain X_l (Fig. 11).

Let $\mathcal{P}'$ be the sequence defined as in Lemma 2, and $\mathcal{P}'_X$ the sequence obtained from $\mathcal{P}'$ by deleting i, X_i, Y_i, Z_i for every $i \in X(\mathcal{P}')$, where $X(\mathcal{P}') = \{i : X_i \in \mathcal{P}(j)$ for some $j \in [2, n-1]\}$. Since $\mathcal{P}(1), \ldots, \mathcal{P}(l-1)$ do not contain X_l, $\mathcal{P}'_X$ contains Y_l or Z_l between 2 and r_m, which implies that the region consisting of room $r_2, \ldots, r_m$ is not a rectangle on the left face of the entire rectangular solid for the 3D-dissection representing by $\mathcal{P}'$ by Theorem 2 and Lemma 4. However, this region must be a rectangle since by Lemma 2, the 3D-dissection represented by $\mathcal{P}'$ is obtained from the original 3D-dissection by sliding the prime wall of room 1 until room 1 is deleted. Hence, we have a contradiction. □

From Theorem 7, we have the following theorem.

Theorem 8: Condition 3 in Theorem 4 is satisfied. □

Theorems 5, 6, and 8 complete the proof of necessity of Theorem 4.

3.1.2 Proof of Sufficiency of Theorem 4

The sufficiency is proved by induction on n.

If $n = 1$ then a P-sequence $\mathcal{P}$ is unique. That is $\mathcal{P} = X_1Y_1Z_1 1$, which represents the O-sequence of the rectangular solid dissection consisting of only one room.

Assume for induction that the sufficiency holds for $n < n'$, and consider a P-sequence

$$\mathcal{P} = \mathcal{P}(W_X)\mathcal{P}(W_Y)\mathcal{P}(W_Z)1\mathcal{P}(1)2\mathcal{P}(2)\cdots\mathcal{P}(n'-1)n',$$

Since $\mathcal{P}$ satisfies Condition 1 in Theorem 4, we can assume without loss of generality that

$$\mathcal{P}(1) = X_{r_m}X_{r_{m-1}}\cdots X_{r_1}.$$

Since $\mathcal{P}$ satisfies Condition 2 in Theorem 4, we can represent

$$\mathcal{P}(W_X) = \mathcal{P}'(W_X)X_1$$
$$\mathcal{P}(W_Y) = \mathcal{P}'(W_Y)Y_1$$
$$\mathcal{P}(W_Z) = \mathcal{P}'(W_Z)Z_1.$$

Let

$$\mathcal{P}' = (\mathcal{P}'(W_X)\mathcal{P}(1))\mathcal{P}'(W_Y)\mathcal{P}'(W_Z)2\mathcal{P}(2)\cdots\mathcal{P}(n-1)n.$$

Since then $\mathcal{P}$ satisfies Conditions 1, 2, and 3 in Theorem 4, $\mathcal{P}'$ also satisfies these conditions, and hence, by inductive hypothesis, $\mathcal{P}'$ is an O-sequence. Let

$$X(\mathcal{P}') = \{i : X_i \in \mathcal{P}(j) \text{ for some } j \in [2, n-1]\},$$

and let $\mathcal{P}'_X$ denote the sequence from $\mathcal{P}'$ by deleting i, X_i, Y_i, and Z_i for every $i \in X(\mathcal{P}')$. Then, by Lemma 4, $\mathcal{P}'_X$ is the Q-sequence of the 2D-dissection on the left face of the entire rectangular solid represented by $\mathcal{P}'$. By Theorem 1 and the definition of $\mathcal{P}'_X$, we can represent

$$\mathcal{P}'_X = \mathcal{P}''(W_Y)\mathcal{P}''(W_Z)$$
$$r_m\mathcal{P}'(r_m)r_{m-1}\mathcal{P}'(r_{m-1})\cdots\mathcal{P}'(r_2)r_1\cdots,$$

where $\mathcal{P}''(W_Y)$ [$\mathcal{P}''(W_Z)$] is a sequence of Ys [Zs]. Since $\mathcal{P}$ satisfies Condition 3 in Theorem 4, we conclude that, for every positive integer $i \leq m$, $\mathcal{P}'(r_i)$ contains neither Y_k nor Z_k with $k > i$, and so, by Theorem 2, the region consisting of rooms $r_m, r_{m-1}, \ldots, r_1$ on the left face is a rectangle. Hence, we can obtain a rectangular solid dissection with n rooms from the rectangular solid dissection represented by $\mathcal{P}'$ by inserting one room from the left of rooms $r_m, r_{m-1}, \ldots, r_1$, and then $\mathcal{P}$ is the O-sequence of this new rectangular solid

```
Decoding Algorithm
    Input: O-sequence 𝒪;
    Output: 3D-dissection;
{
  Prepare an entire rectangular solid dissected
               into only one room, labeled with n;
  for (k = n − 1, n − 2, ..., 1) {
    if (𝒪(k) = X_{r_m} X_{r_{m−1}} ··· X_{r_1})
      Insert room k from the left of
                    rooms r_m, r_{m−1}, ..., r_1;
    if (𝒪(k) = Y_{r_m} Y_{r_{m−1}} ··· Y_{r_1})
      Insert room k from the front of
                    rooms r_m, r_{m−1}, ..., r_1;
    if (𝒪(k) = Z_{r_m} Z_{r_{m−1}} ··· Z_{r_1})
      Insert room k from the top of
                    rooms r_m, r_{m−1}, ..., r_1;
  }
}
```

Fig. 12 Algorithm for decoding O-sequence.

dissection, which completes the proof of the sufficiency. □

Notice that the above proof of the sufficiency presents the decoding algorithm for an O-sequence in Fig. 12. This algorithm perfomes in $O(n)$ time.

3.2 Checking the Feasibility of a Code

The time complexity of checking the feasibility of a code is $O(n)$. It is easy to decide whether a given code $\mathcal{P}$ satisfies Conditions 1 and 2 in Theorem 4. It is also easy to see that the algorithm in Fig. 13 determines whether $\mathcal{P}$ satisfies Condition 3 in $O(n)$ time. The stack S_x[resp. S_y and S_z] in the algorithm stores the simplified Q-Sequence of 2D-dissection on the left[front and top] face of the entire rectangular solid. If the room i is not a rectangular solid after inserted into the entire rectangular solid, $\mathcal{P}$ does not satisfy Condition 3.

4. Size of a Solution Space of O-Sequence

The number of 3D-dissection which is dissected by only rectangular and non-crossing walls is presented in [9]. Since there is an one-to-one corresponding relation between an O-sequence and a 3D-dissection which is dissected by only rectangular and non-crossing walls, the number of codes of O-sequence is equal to the number of 3D-dissection which is dissected by only rectangular and non-crossing walls.

A size of solution space of 3D-CBL is less than $n!3^{n-1}2^{4n-4}$†. However, the exact size of a solution space of 3D-CBL is not written in [6].

3D-CBL can represent any 3D-dissection which is dissected by only rectangular and non-crossing walls. However, there is not an one-to-one corresponding relation between a 3D-CBL and a 3D-dissection. Therefore, the size of a solution space of 3D-CBL is larger than that of O-sequence for the following reasons.

```
Determining Whether a Code Satisfies
                         Condition 3 in Theorem 4
   Input: P-sequence 𝒫 which satisfies
                         Conditions 1 and 2 in Theorem 4;
   Output: Whether 𝒫 satisfies
                         Condition 3 in Theorem 4 or not;
{
   Prepare stacks S_x, S_y, and S_z;
   push n to S_x; push n to S_y; push n to S_z;
   for (i = n − 1, n − 2, ..., 1) {
     if (𝒪(i) = X_{r_m} X_{r_{m−1}} ··· X_{r_1}){
       for (j = 1, 2, ..., m){
         if(r_j exists in stack S_y)push X_{r_j} to S_y;
         if(r_j exists in stack S_z)push X_{r_j} to S_z;
       }
       A = pop(S_x);
       while (A ≠ r_m){
         if(A is symbol Y or Z,
             and its subscript is larger than r_m)
           {return 𝒫 does not satisfy Condition 3;}
         A = pop(S_x);
       }
     }
     if (𝒪(i) = Y_{r_m} Y_{r_{m−1}} ··· Y_{r_1}){
       for (j = 1, 2, ..., m){
         if(r_j exists in stack S_z)push Y_{r_j} to S_z;
         if(r_j exists in stack S_x)push Y_{r_j} to S_x;
       }
       A = pop(S_y);
       while (A ≠ r_m){
         if(A is symbol Z or X,
             and its subscript is larger than r_m)
           {return 𝒫 does not satisfy Condition 3;}
         A = pop(S_y);
       }
     }
     if (𝒪(i) = Z_{r_m} Z_{r_{m−1}} ··· Z_{r_1}){
       for (j = 1, 2, ..., m){
         if(r_j exists in stack S_x)push Z_{r_j} to S_x;
         if(r_j exists in stack S_y)push Z_{r_j} to S_y;
       }
       A = pop(S_z);
       while (A ≠ r_m){
         if(A is symbol X or Y,
             and its subscript is larger than r_m)
           {return 𝒫 does not satisfy Condition 3;}
         A = pop(S_z);
       }
     }
     push i to S_x; push i to S_y; push i to S_z;
   }
   return 𝒫 satisfies Condition 3;
}
```

Fig. 13 Algorithm for checking whether P-sequence $\mathcal{P}$ satisfies condition 3 in Theorem 4.

1. Several codes are decoded into the same 3D-dissection, for example, both 3D-CBLs: $S = \{1,2,3\}$, $L = \{X,X\}$, $T = \{1010\}$ and $S = \{1,2,3\}$, $L = \{X,X\}$, $T = \{110\}$ are decoded into the same 3D-dissection shown in Fig. 14.
2. Several other codes are not decoded into a 3D-dissection. For example, 3D-CBL: $S = \{1,2,3,4\}$, $L = \{X,Z,Y\}$, $T = \{10110110\}$ is not corresponding to 3D-dissection. It is corresponding to an object shown in Fig. 15.

The size of the solution space of each representation of 3D-dissections or 3D-packings is shown in Fig. 16, where that of 3D-subTCG [2] is not shown since the number of

†In the literature [6], it is written as $O(n!3^{n-1}2^{4n-4})$.

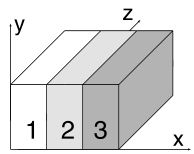

Fig. 14 A result of decoding from 3D-CBL: $S = \{1, 2, 3\}$, $L = \{X, X\}$, $T = \{1010\}$ or $S = \{1, 2, 3\}$, $L = \{X, X\}$, $T = \{110\}$.

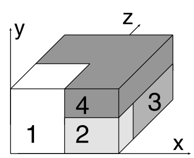

Fig. 15 A result of decoding from 3D-CBL: $S = \{1, 2, 3, 4\}$, $L = \{X, Z, Y\}$, $T = \{10110110\}$. It is not 3D-dissection because a shape of room 4 is not a rectangular solid.

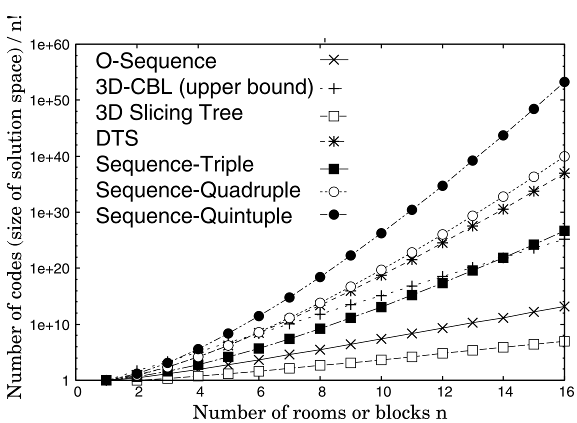

Fig. 16 Comparison of the number of codes: O-sequence, 3D-CBL [6], 3D slicing tree [5], DTS [4], Sequence-Triple [1], Sequence-Quadruple [3], and Sequence-Quintuple [1].

its codes is not clear. Note that vertical axis is the number of codes divided by n! The size of the solution space of O-sequence is the second smallest.

5. Experimental Result

The O-sequence is usable to search 3D-dissections. We search the 3D-dissections, each of rooms contains exactly one block, with simulated annealing. Each room of 3D-dissection is large enough to contain the assigned block, and the volume of an entire rectangular solid of a 3D-dissection is the minimum one. An objective of the search is to find a placement of n blocks with smaller volume. In each iteration of simulated annealing, the algorithm constructs a representation (3D-dissection and directions of blocks) from the current representation of rectangular solids by performing one of the following three operations: (1) Exchange of the blocks in a pair of rooms; (2) Rotating of a block; (3) Deletion and insertion of a room. The details of deletion and insertion are described in [10].

The algorithm was implemented in C language on a PC with Pentium4 3.2 GHz. Fig. 17 shows a packing result for

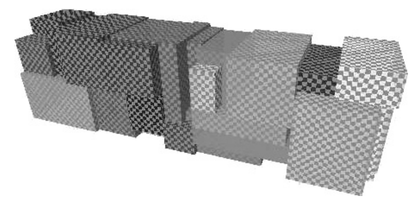

Fig. 17 Resultant placement with simulated annealing (Time: 375 [sec]. volume ratio: 113%).

30 blocks made at random, which was obtained in 375 seconds. The ratio of the volume of the entire rectangular solid to the total volume of the blocks (volume ratio) is 113%.

6. Conclusion

In this paper, we proposed a string data structure, called O-sequence, to represent any 3D-dissection which is dissected by only non-crossing rectangular walls. There is one-to-one corresponding relation between an O-Sequence and a 3D-dissection. We presented a necessary and sufficient condition for a given sequence to be an O-sequence, and also presented an algorithm for decoding an O-sequence into the corresponding 3D-dissection, which performs in $O(n)$ time.

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